International Journal of Mathematics, Statistics and Operations Research
Vol. 3; Number 1; 2023, pp. 27-39
https://doi.org/10.47509/IJMSOR.2023.v03i01.02

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

New Properties of Higher-Order Radial Sets and Higher-Order Radial Derivatives and Applications to Optimality Conditions[1]

Tao Du, Qilin Wang[2]
College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing, 400074, China
E-mails: dutao202209@163.com, wangql97@126.com

Abstract: Several new properties are obtained for higher-order upper radial sets and corresponding derivatives introduced in [J. Glob. Optim. DOI 10.1007/s10898-012-9861-z]. Higher-order sufficient optimality conditions and higher-order necessary optimality conditions are established for weak efficient solutions of set-valued optimization problems. Several corollaries are provided to show that obtained results include a few existing ones. In addition, one removes deficiencies contained of two results in [J. Glob. Optim. DOI 10.1007/s10898-012-9861-z].

Keywords: Higher-order upper radial sets, Higher-order upper radial derivatives, Set-valued vector optimization, Weak efficiency, Higher-order optimality conditions.

[1]This research was partially supported by This research was partially supported by the Group Building Project for Scientific Innovation for Universities in Chongqing (CXQT21021) and the Joint Training Base Construction Project for Graduate Students in Chongqing (JDLHPYJD2021016).

[2]Corresponding author.

Article History
Received : 09 January 2023; Revised : 04 February 2023; Accepted : 11 February 2023; Published : 22 May 2023

To cite this paper
Tao Du, Qilin Wang (2023). New Properties of Higher-Order Radial Sets and Higher-Order Radial Derivatives and Applications to Optimality Conditions. *International Journal of Mathematics, Statistics and Operations Research.* 3(1), 27-39.

1 Introduction

In set-valued analysis, the notion of a derivative of a set-valued map has been formulated in different ways and applied to set up the optimality conditions(see [1–11]) . By virtue of the concept of contingent derivative for a set-valued map (see [2]), Corley [6] investigated optimality conditions for set-valued optimization problems. But it turns out that necessary and sufficient optimality conditions do not coincide under standard assumptions. For solving this problem, Jahn and Rauh [7] introduced the contingent epiderivative of a set-valued map and then obtained unified necessary and sufficient optimality conditions. But, unfortunately, since the contingent epiderivative of a set-valued map is a single-valued map, the conditions assuring the existence of the contingent epiderivative are hard to be satisfied. To overcome the difficulty, Chen and Jahn [8] introduced a generalized contingent epiderivative of a set-valued map, and established a unified necessary and sufficient optimality condition for set-valued optimization problems in terms of the generalized contingent epiderivative. Li and Chen [9] proposed higher-order generalized contingent(adjacent) epiderivatives of set-valued maps, and then obtained higher-order Fritz John type necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem. Li et al. [10] studied some properties of higher-order tangent sets and higher-order derivatives introduced in [2], and then obtained higher-order necessary and sufficient optimality conditions for set-valued optimization problems under cone-concavity assumptions. In [11], Li et al. introduced generalized second-order composed contingent epiderivatives for set-valued maps and established a unified sufficient and necessary optimality condition for set-valued optimization problems by employing the generalized second-order composed contingent epiderivatives. In [12], using the concept of the radial epiderivatives, Kasimbeyli obtained necessary and sufficient optimality conditions for optimization problems without convexity conditions . By employing higher-order upper radial set and higher-order upper radial derivative, Anh et al. [13] established optimality conditions of weakly efficient solutions for set-valued optimization problems. Wang et al. [14] proposed the higher-order weak radial epiderivative of a set-valued map, and obtained the optimal- ity conditions for non-convex set-valued optimization problems under the weakly efficiency. Zhang and Wang [15] introduced the second-order weakly composed radial epiderivative of set- valued maps, and obtained the necessary optimality conditions of Benson proper efficient solutions for the constrained set-valued optimization problems without the assumptions of generalized cone-convexity. Peng et al. [16] provided the higher-order weak lower inner Studniarski epiderivative for set- valued maps, and obtained KarushCKuhnCTucker necessary optimality conditions for Benson proper efficient solutions of the constrained set-valued optimization problems.

Motivated by the work reported in [10–16], several new properties are obtained for higher-order upper radial sets and higher-order upper radial derivatives introduced in [13], and by virtue of the properties, a few optimality conditions are established for weak efficient solutions of a set-valued optimization problem. some recent existing results are derived from the obtained ones. In addition, one removes deficiencies contained in two earlier results in [13].

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts. In Section 3, we obtain several properties of the higher-order upper radial sets and higher-order upper radial derivatives. In Section 4, we establish optimality conditions for weak efficient solutions of constrained set-valued optimization problems.

2 Preliminaries

Throughout this paper, if not otherwise specified, let X, Y and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with $intC \neq \emptyset$ and $intD \neq \emptyset$, respectively. one assumes that $0_X, 0_Y, 0_Z$ denote the origins of X, Y, Z, respectively, Y^* denotes the topological dual space of Y and C^* denotes the dual cone of C, defined by $C^* = \{\varphi \in Y^* | \varphi(y) \geq 0, \forall y \in C\}$. Let S be a nonempty subset of X, $F : S \to 2^Y$ and $G : S \to 2^Z$ be two given nonempty set-valued maps. The graph of F is defined by $\mathrm{gr}F = \{(x, y) \in X \times Y | x \in S, y \in F(x)\}$. The profile map $F_+ : S \to 2^Y$ is defined by $F_+(x) = F(x) + C$, for every $x \in S$.

Definition 2.1 [2] *Let M be a nonempty subset in X, $\breve{x} \in M$ We say that $\breve{x}$ is a C-weakly efficient point of M if*

$$(M - \{\breve{x}\}) \cap (-\mathrm{intC}) = \emptyset.$$

Definition 2.2 (See [17]) Let $S \subseteq X$ be a nonempty subset and $x_0 \in \mathrm{cl}S$. The closed radial cone $T^r_S(x_0)$ to S at x_0 is the set of all $v \in X$ for which there exist a sequence $\{\lambda_n\}$ of positive real numbers and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} x_n = v$ such that $x_0 + \lambda_n x_n \in S$, for all $n \in N$, where N denotes the natural number set.

Definition 2.3 (See [17]) Let $S \subseteq X$ be a nonempty subset, $F : S \to Y$ be a set-valued map and $(x_0, y_0) \in \text{gr}F$. The radial derivative $D^r F(x_0, y_0)$ of F at (x_0, y_0) is a set-valued map from X to Y defined by

$$y \in D^r F(x_0, y_0)(x) \text{ if and only if } (x, y) \in T^r_{\text{gr}F}(x_0, , y_0).$$

Definition 2.4 (See [13]) Let $x \in S \subseteq X$, $F : S \to 2^Y$, $(x_0, y_0) \in \text{gr}F$ and $(u_1, v_1), \cdots,$ $(u_{m-1}, v_{m-1}) \in X \times Y$ with $m \geq 1$.

(i) The mth-order upper radial set of S with respect to $u_1, \cdots, u_{m-1}$ is defined as

$$T^{r(m)}_S(x, u_1, \cdots, u_{m-1})$$
$$= \{x \in X | \exists t_n > 0, \exists x_n \to x, \forall n, x_0 + t_n u_1 + \cdots + t_n^{m-1} u_{m-1} + t_n^m x_n \in S\}.$$

(ii) The mth-order upper radial derivative of F at (x_0, y_0) with respect to $(u_1, v_1), \cdots,$ (u_{m-1}, v_{m-1}) is the set-valued map $D^{(m)}_R F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}) : X \to 2^Y$ whose graph is

$$grD^{(m)}_R F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}) = T^{r(m)}_{\text{gr}F}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}).$$

From the definition, one knows that the following results hold.

Proposition 2.1 Let $x \in S \subseteq X$, $F : S \to 2^Y$ and $(x_0, y_0) \in \text{gr}F$. Then

(i) $T^{r(m)}_S(x, 0_X, \cdots, 0_X) = T^r_S(x)$;

(ii) $D^m_R F(x_0, y_0, 0_X, 0_Y, \cdots, 0_X, 0_Y) = D^r F(x_0, y_0)$.

Proposition 2.2 *(see [13, Proposition3.4])* Let $S = domF$ and $(x_0, y_0) \in \text{gr}F$. Then, for all $x \in S$,

(i) $F(x) - \{y_0\} \subseteq D^r F(x_0, y_0)(x - x_0)$;

(ii) $F(x) - \{y_0\} \subseteq T^r_{F(S)}(y_0)$.

Take $x = x_0$ in Proposition 2.2, one obtains the following results.

Corollary 2.1 Let $S = domF$ and $(x_0, y_0) \in \text{gr}F$. Then

(i) $0_Y \in D^r F(x_0, y_0)(0_X)$;

(ii) $0_Y \in T^r_{F(S)}(y_0)$.

In this paper, consider the following set-valued vector optimization problem:

$$(P)\begin{cases} min & F(x), \\ s.t. & x \in S, G(x)\bigcap(-D) \neq \emptyset. \end{cases}$$

The point $(x_0, y_0) \in \mathrm{gr}F$ is said to be a weakly efficient solution of problem (P) if

$$(F(A) - \{y_0\}) \cap (-\mathrm{int}D) = \emptyset,$$

where $A :=\in \{x \in S | G(x)\bigcap(-D) \neq \emptyset\}$.

For other notations and definitions, one refers to Ref. [13].

3 Properties of Higher-Order Upper Radial Sets and Upper Radial Derivatives

Proposition 3.1 Let $x \in S \subseteq X$, $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $u_i = 0_X \in X, v_i \in -C, i = 1, \cdots, m-1$. Then

$$T^{r(m)}_{F_+(S)}(y_0, v_1, \cdots, v_{m-1}) \subset T^r_{F_+(S)}(y_0), \tag{1}$$

and

$$D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \subset D^r F_+(x_0, y_0)(x), \forall x \in X. \tag{2}$$

Proof. We first prove that (1) holds.

Let $y \in T^{r(m)}_{F_+(S)}(y_0, v_1, \cdots, v_{m-1})$. Then there exist sequences $t_n > 0$, $x_n \in S$ and $y_n \in F(x_n) + C$ such that

$$\frac{y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1} v_{m-1}}{t_n^m} \to y. \tag{3}$$

Since $y_n \in F(x_n) + C$, $v_i \in -C, i = 1, \cdots, m-1$ and $t_n > 0$, $\overline{y}_n := y_n - t_n v_1 - \cdots - t_n^{m-1} v_{m-1} \in F(x_n) + C$. Combined with (16), we can conclude that

$$\frac{\overline{y}_n - y_0}{t_n^m} \to y,$$

which implies $y \in T^r_{F_+(S)}(y_0)$. Thus (1) holds.

we next prove that (2) holds. Let $x \in X$. Let us consider two possible cases for $D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$.

Case 1: $D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) = \emptyset$. (2) holds trivially.

Case 2: $D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \neq \emptyset$. Let $y \in D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$. Then there exist sequences $t_n > 0$, $x_n \in S$ and $y_n \in F(x_n) + C$ such that

$$\frac{(x_n, y_n) - (x_0, y_0) - t_n(u_1, v_1) - \cdots - t_n^{m-1}(u_{m-1}, v_{m-1})}{t_n^m} \to (x, y). \tag{4}$$

Set $\overline{x}_n := x_n - t_n u_1 - \cdots - t_n^{m-1} u_{m-1}$. Since $y_n \in F(x_n) + C$, $u_i = 0_X, v_i \in -C, i = 1, \cdots, m-1$ and $t_n > 0$,

$$\overline{y}_n := y_n - t_n v_1 - \cdots - t_n^{m-1} v_{m-1} \in F(\bar{x}_n) + C.$$

Combined with (4), we can conclude that

$$\frac{(\overline{x}_n, \overline{y}_n) - (x_0, y_0)}{t_n^m} \to (x, y),$$

which implies $y \in D^r F_+(x_0, y_0)(x)$. Thus (2) holds and the proof is complete. □

By Proposition 3.1 and [13, Remark 3.2(iv)], we have the following result.

Corollary 3.1 Let $x \in S \subseteq X$, $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $u_i \in X, v_i \in -C, i = 1, \cdots, m-1$. Then

$$D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(X) \subset T^r_{F_+(S)}(y_0).$$

Proposition 3.2 Let $x \in S \subseteq X$, $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $u_i = 0_X, v_i \in C, i = 1, \cdots, m-1$. Then

$$F(S) - \{y_0\} \subset T^{r(m)}_{F_+(S)}(y_0, v_1, \cdots, v_{m-1}),$$

$$F(x) - \{y_0\} \subset D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0).$$

Proof. Let $x \in S$ and $y \in F(x)$. Take $x_n = x$, $y_n = y + v_1 + v_2 + \cdots + v_{m-1}$ and $t_n = 1$. Since $v_i \in C, i = 1, \cdots, m-1$, $y_n \in F(x_n) + C$. Thus, it follows from the definitions of mth-order upper radial sets that

$$\frac{(x_n, y_n) - (x_0, y_0) - t_n(u_1, v_1) - \cdots - t_n^{m-1}(u_{m-1}, v_{m-1})}{t_n^m} = (x - x_0, y - y_0)$$

$$\in T^{r(m)}_{\mathrm{gr}F_+}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1}),$$

which implies

$$y - y_0 \in T^{r(m)}_{F_+(S)}(y_0, v_1, \cdots, v_{m-1}),$$

$$y - y_0 \in D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0).$$

So $F(S) - \{y_0\} \subset T^{r(m)}_{F_+(S)}(y_0, v_1, \cdots, v_{m-1})$ and $F(x) - \{y_0\} \subset D^m_R F_+(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x - x_0)$. The proof is complete. □

4 Optimality Conditions

In this section, we present optimality conditions for weakly efficient solutions of set-valued optimization problems. In addition, we remove deficiencies contained in two earlier results in [13].

Theorem 4.1 Let $(x_0, y_0) \in \mathrm{gr}F$ and $z_0 \in G(x_0) \bigcap (-D)$. If (x_0, y_0) is a weakly efficient pair of (P), then, the following separations holds

$$T^r_{(F,G)_+(S)}(y_0, z_0) \bigcap -int(C \times D) = \emptyset, \tag{5}$$

and

$$D^r(F, G)_+(x_0, y_0, z_0)(X) \bigcap -int(C \times D) = \emptyset. \tag{6}$$

Proof. By Proposition 2.1 and [13, Remark 3.2(iv)], we can easily derive (6) from (5). Therefore, we need to prove only that (5) holds. Suppose that (5) does not hold. Then, there exists $(y, z) \in Y \times Z$ such that

$$(y, z) \in T^r_{(F,G)_+(S)}(y_0, z_0) \tag{7}$$

and

$$(y, z) \in -int(C \times D). \tag{8}$$

It follows from (7) and the definition of first-order upper radial sets that there exist sequences $t_n > 0, x_n \in S$ and $(y_n, z_n) \in (F, G)(x_n) + C \times D$ such that

$$\frac{(y_n, z_n) - (y_0, z_0)}{t_n} \to (y, z). \tag{9}$$

From (8),(9) and $z_0 \in -D$, there exists large enough natural number N such that

$$y_n - y_0 \in -intC,\ z_n \in -intD,\ \forall n > N. \tag{10}$$

Since $z_n \in G(x_n) + D$, there exist $\overline{z}_n \in G(x_n)$ and $d_n \in D$ such that $z_n = \overline{z}_n + d_n$. It follows from (10) that $\overline{z}_n \in G(x_n)\bigcap(-D), \forall n > N$, which implies $x_n \in A$, for any $n > N$. Since $y_n \in F(x_n) + C$, there exist $\overline{y}_n \in F(x_n)$ and $c_n \in C$ such that $y_n = \overline{y}_n + c_n$. It follows from (10) that

$$\overline{y}_n - y_0 \in (F(x_n) - \{y_0\})\bigcap(-intC) \subset (F(A) - \{y_0\})\bigcap(-intC), \forall n > N,$$

which contradicts that (x_0, y_0) be a weakly efficient pair of (P). So (5) holds and the proof is complete. □

Remark 4.1 *By Proposition 3.1 and Corollary 3.1, we can easily derive [13, Theorem 4.1] from Theorem 4.1.*

Corollary 4.1 (see [13, Theorem 4.1]) Let $(x_0, y_0) \in \mathrm{gr}F$ be a weakly efficient pair of (P), $z_0 \in G(x_0)\bigcap(-D)$, $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \cdots, m-1$. Then, the following separations holds

$$T^{r(m)}_{(F,G)+(S)}((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1}))\bigcap -int(C \times D) = \emptyset$$

and

$$D^m_R(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(X)\bigcap -int(C \times D) = \emptyset.$$

Theorem 4.2 Let $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0)\bigcap(-D)$, $(u_i, v_i, w_i) \in \{0_X\} \times C \times D, i = 1, 2, \cdots, m-1$. If one of the following separations holds

(i)

$$T^{r(m)}_{(F,G)+(S)}((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1}))\bigcap -(intC \times D(z_0)) = \emptyset, \tag{11}$$

(ii)

$$D^m_R(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0)$$

$$\bigcap -(intC \times D(z_0)) = \emptyset, x \in A, \tag{12}$$

then (x_0, y_0) is a weakly efficient pair of (P).

Proof. We need to prove only that (i) holds. It follows from Proposition 3.2 that

$$(F,G)(x) - \{(y_0, z_0)\} \subset T^{r(m)}_{(F,G)+(S)}((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1})), \forall x \in A.$$

Thus, by (11), we have

$$[(F,G)(x) - \{(y_0, z_0)\}] \bigcap -(intC \times D(z_0)) = \emptyset, \forall x \in A. \quad (13)$$

Suppose that there exist $x \in A$ and $y \in F(x)$ such that $y - y_0 \in -intC$. Then there exists $z \in G(x) \bigcap(-D)$ such that $z - z_0 \in -D(z_0)$, and hence

$$(y, z) - (y_0, z_0) \in -(intC \times D(z_0)),$$

which contradicts (13). So (x_0, y_0) is a weakly efficient pair of (P) and the proof is complete. □

Corollary 4.2 Let $(x_0, y_0) \in \text{gr}F$ and $z_0 \in G(x_0) \bigcap(-D)$. If one of the following separations holds

(i) $T^r_{(F,G)+(S)}((y_0, z_0)) \bigcap -(intC \times D(z_0)) = \emptyset$,

(ii) $D^r(F,G)_+(x_0, y_0, z_0)(x - x_0) \bigcap -(intC \times D(z_0)) = \emptyset, x \in A$,

then (x_0, y_0) is a weakly efficient pair of (P).

Remark 4.2 (i)Since [13, (8)] need be satisfied for any vector group $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \cdots, m-1$, and equalities (11) and (12) need be satisfied for a vector group $(u_i, v_i, w_i) \in \{0_X\} \times C \times D, i = 1, 2, \cdots, m-1$, Theorem 4.2 improves [13, Theorem 4.4].
(ii) Take $G(x) \equiv Z$. Then [12, Theorem 4.4] can be obtained from Corollary 4.2 and Theorem 4.1.
(iii) By Corollary 4.2 and the proof of [13, Theorem 4.4], [13, Theorem 4.4] can be derived from Theorem 4.2.

Corollary 4.3 (see [13, Theorem 4.4]) Let $(x_0, y_0) \in \text{gr}F$. Suppose that there exists $z_0 \in G(x_0) \bigcap(-D)$ such that, for $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \cdots, m-1$, and x in the feasible set A, one of the following separations holds

(i) $T^{r(m)}_{(F,G)+(S)}((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1})) \bigcap -(intC \times D(z_0)) = \emptyset$,

(ii) $D_R^m(F,G)_+(x_0,y_0,z_0,u_1,v_1,w_1,\cdots,u_{m-1},v_{m-1},w_{m-1})(x-x_0)\bigcap-(intC\times D(z_0))=\emptyset$.

Then, (x_0,y_0) is a weakly efficient pair of (P).

By employing mth-order upper radial set and mth-order upper radial derivative, Anh et al.(see [13]) established the following sufficient optimality conditions of weakly efficient solutions for (P):

Theorem A (see [13, Theorem 4.5]) Let the assumptions of [13, Theorem 4.4] be satisfied. Then, (x_0,y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) For all $(y,z)\in T^{r(m)}_{(F,G)_+}(A)((y_0,z_0),(v_1,w_1),\cdots,(v_{m-1},w_{m-1}))$, there exists $(c^*,d^*)\in C^*\times D^*\setminus\{0,0\}$ such that $\langle d^*,z_0\rangle=0$ and

$$\langle c^*,y\rangle+\langle d^*,z\rangle>0. \tag{14}$$

(ii) For all $x\in A$ and all $(y,z)\in D_R^m(F,G)_+(x_0,y_0,z_0,u_1,v_1,w_1,\cdots,u_{m-1},v_{m-1},w_{m-1})(x-x_0)$, there exists $(c^*,d^*)\in C^*\times D^*\setminus\{0,0\}$ such that $\langle d^*,z_0\rangle=0$ and

$$\langle c^*,y\rangle+\langle d^*,z\rangle>0. \tag{15}$$

Theorem B (see [13, Theorem 4.6]) For problem (P), $(x_0,y_0)\in \mathrm{gr}F$, $z_0\in G(x_0)\bigcap(-D)$. Let $(e,k)\in int(C\times D)$. Then, (x_0,y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) There exists $(\Gamma,L)\subset C^*\times D^*\setminus\{(0,0)\}$ such that

$$C=\{y\in Y|\langle f,y\rangle\geq 0, for\ any\ f\in\Gamma\}, D=\{z\in Z|\langle g,z\rangle\geq 0, for\ any\ g\in L\},$$

$$\sup_{(f,g)\in(\Gamma,L)}\{\frac{\langle f,0_Y\rangle+\langle g,-z_0\rangle}{\langle f,e\rangle+\langle g,k\rangle}\}=0, \tag{16}$$

and

$$\sup_{(f,g)\in(\Gamma,L))}\{\frac{\langle f,y\rangle+\langle g,z\rangle}{\langle f,e\rangle+\langle g,k\rangle}\}>0 \tag{17}$$

for any $(y,z)\in T^{r(m)}_{(F,G)_+(A)}((y_0,z_0),(v_1,w_1),\cdots,(v_{m-1},w_{m-1}))$.

(ii) (16) and (17) satisfy for all $(y,z)\in D_R^m(F,G)_+(x_0,y_0,z_0,u_1,v_1,w_1,\cdots,u_{m-1},v_{m-1},w_{m-1})(x-x_0)$ for each $x\in A$.

Unfortunately, [13, Theorems 4.5 and 4.6] exist gaps. On the one hand, from the proofs of [13, Theorems 4.4 and 4.6], it is possible that $u_i = 0_Y$ and $v_i = 0_Z, i = 1, 2, \cdots, m-1$ in [13, Theorems 4.5 and 4.6], and then, the assumptions of [13, Theorems 4.5 and 4.6] should be satified for the case. In fact, by Propposition 2.1 and Corollary 2.1, for any $(x, y) \in grF, z \in G(x)$, one obtains $(0_Y, 0_Z) \in T^{r(m)}_{(F,G)_+(A)}(y, z, 0_Y, 0_Z, \cdots, 0_Y, 0_Z) = T^r_{(F,G)_+(A)}(y, z)$ and $(0_Y, 0_Z) \in D^m_R(F, G)_+(x, y, z, 0_X, 0_Y, 0_Z, \cdots, 0_X, 0_Y, 0_Z)(0_X)$. Therefore, for any $(\Gamma, L) \subset (C^* \times D^*) \setminus (0_{Y^*}, 0_{Z^*})$, the conditions (14), (15) and (17) never hold. On the other hand, the condition (16) can be simply written as

$$z_0 = 0_Z. \tag{18}$$

Indeed, (18)$\Rightarrow$ (16) is obvious. In what concerns the implication (16)$\Rightarrow$ (18), it follows from $z_0 \in -D$ that $g(-z_0) \geq 0$, for all $g \in L \subset D^+$. Thus, if (16) holds, then for all $(f, g) \in \Gamma \times L$, we have

$$0 \leq \frac{\langle f, 0_Y\rangle + \langle g, -z_0\rangle}{\langle f, e\rangle + \langle g, k\rangle} \leq \sup_{(f',g')\in\Gamma\times L} \{\frac{\langle f', 0_Y\rangle + \langle g', -z_0\rangle}{\langle f', e\rangle + \langle g', k\rangle}\} = 0,$$

which implies $g(-z_0) = 0$, for all $g \in L$. This means that $-z_0, z_0 \in \{x \in Z | g(x) \geq 0, \forall g \in L\} = D$. Since D is pointed, one concludes that $z_0 = 0_Z$.

We next give Theorems 4.3 and 4.4 which are appropriate modifications for deficiencies contained in [13, Theorems 4.5 and 4.6].

Theorem 4.3 Let the assumptions of [13, Theorem 4.4] be satisfied. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) For any $(y, z) \in (T^{r(m)}_{(F,G)_+}(A)((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1})) \setminus \{(0_Y, 0_Z)\}$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle d^*, z_0\rangle = 0$ and $\langle c^*, y\rangle + \langle d^*, z\rangle > 0$.

(ii) For each $x \in A$ and all $(y, z) \in D^m_R(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0)$ with $(y, z) \neq (0_Y, 0_Z)$, there exists $(c^*, d^*) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle d^*, z_0\rangle = 0$ and $\langle c^*, y\rangle + \langle d^*, z\rangle > 0$.

Theorem 4.4 Let $(x_0, y_0) \in \mathrm{gr}F$, $z_0 \in G(x_0) \bigcap(-D)$ and $(e, k) \in int(C \times D)$. Then, (x_0, y_0) is a weakly efficient pair of (P) if one of the following conditions holds.

(i) $z_0 = 0_Z$ and there exists $(\Gamma, L) \subset (C^* \times D^*) \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $C = \{y \in Y | \langle f, y\rangle \geq 0, for\ any\ f \in \Gamma\}, D = \{z \in Z | \langle g, z\rangle \geq 0, for\ any\ g \in L\}$,

$$\sup_{(f,g)\in(\Gamma,L))} \{\frac{\langle f, y\rangle + \langle g, z\rangle}{\langle f, e\rangle + \langle g, k\rangle}\} > 0, \tag{19}$$

for any $(y, z) \in T^{r(m)}_{(F,G)_+(A)}((y_0, z_0), (v_1, w_1), \cdots, (v_{m-1}, w_{m-1})) \setminus \{(0_Y, 0_Z)\}$.

(ii) $z_0 = 0_Z$ and (19) satisfy for all $(y, z) \in D^m_R(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0) \setminus \{(0_Y, 0_Z)\}$, for each $x \in A$.

References

[1] Jiménez, B. and Novo, V.(2004),"Optimality conditions in differentiable vector optimization via second-order tangent sets", Appl. Math. Optim. 49, 123-144.

[2] Aubin, J. P. and Frankowska, H.(1990), 'Set-Valued Analysis, Biekhäuser: Boston.

[3] Luc, D. T.(1989), Theory of vector Optimization, Springer, Berlin.

[4] Jahn, J.(2004), Vector Optimization: Theory, Applications, and Extensions, Springer, Berlin.

[5] Jiménez, B. and Novo V.(2003),"Second-order necessary conditions in set constrained differentiable vector optimization", Math. Methods Oper. Res. 58, 299-317.

[6] Corley, H. W.(1988), "Optimality condition for maximizations of set-valued functions", J. Optim. Theory Appl. 58, 1-10.

[7] Jahn, J., Rauh, R.(1997), "Contingent epiderivatives and set-valued optimization", Math. Methods Oper. Res. 46, 193-211.

[8] Chen, G. Y. and Jahn, J.(1998), "Optimality conditions for set-valued optimization problems", Math. Methods Oper. Res. 48 , 187-200.

[9] Li, S. J. and Chen, C. R.(2006), "Higher-order optimality conditions for Henig efficient solutions in set-valued optimization", J. Math. Anal. Appl. 323, 1184-1200.

[10] Li, S. J., Teo, K. L. and Yang, X. Q.(2008), "Higher-order optimality conditions for set-valued optimization", J. Optim. Theory Appl. 137(3),533-553.

[11] Li, S. J., Zhu, S. K. and Teo, K. L.(2012), "New generalized second-order contingent epiderivatives and set-valued optimization problems", J. Optim. Theory Appl. 152(3),587-604.

[12] Kasimbeyli, R.(2009), “Radial epiderivatives and set-valued optimization ”, Optimization. 58(5), 521-534.

[13] Anh, N. L. H. and Khanh, P. Q.(2013), “Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives”, J. Global Optim. 56(2):519-536.

[14] Wang, Q.L., He, L. and Li, S.J.(2019), “Higher-order weak radial epiderivatives and non-convex set-valued optimization problems”, J. Ind. Manag. Optim. 15, 465C480.

[15] Zhang, X.Y. and Wang, Q.L.(2020), “New second-order radial epiderivatives and applications to optimality conditions”, Rairo-Oper. Res. 54, 949-959.

[16] Peng, Z.H., Wan, Z.P. and Guo, Y.J.(2020), “New higher-order weakly lower inner epiderivatives and application to KarushCKuhnCTucker necessary optimality conditions in set-valued optimization”, Japan J. Indust. Appl. Math. J. 37, 851-866.

[17] Taa, A.(1998), “Set-valued derivatives of multifunctions and optimality conditions”, Numer. Funct. Anal. Optim. 19,121-140.